ARBITRARY RANK JUMPS FOR A-HYPERGEOMETRIC SYSTEMS THROUGH LAURENT POLYNOMIALS

LAURA FELICIA MATUSEVICH AND ULI WALTHER

ABSTRACT. We investigate the solution space of hypergeometric systems of differential equations in the sense of Gelfand, Graev, Kapranov and Zelevinsky. For any integer $d \geq 2$ we construct a matrix $A_d \in \mathbb{N}^{d\times 2d}$ and a parameter vector β_d such that the holonomic rank of the A-hypergeometric system $H_{A_d}(\beta_d)$ exceeds the simplicial volume $\mathrm{vol}(A_d)$ by at least $d-1$. The largest previously known gap between rank and volume was two.

Our argument is elementary in that it uses only linear algebra, and our construction gives evidence to the general observation that rank-jumps seem to go hand in hand with the existence of multiple Laurent (or Puiseux) polynomial solutions.

1. INTRODUCTION

A power series $\sum_{t=1}^{\infty} a(t)x^t$ is geometric, if the assignment $t \mapsto a(t+1)/a(t)$ is a constant function on $\mathbb{N}$. If the value of these quotients is always λ, then clearly $a(t) = c\cdot\lambda^t$ for some constant c. A natural generalization are the *hypergeometric series* for which $a(t+1)/a(t)$ is a rational function in t. The study of such objects goes back at least to Euler. Gauß continued this work and Kummer and Riemann pioneered the idea of investigating the differential equations that are satisfied by a given hypergeometric series.

Hypergeometric differential equations and their solutions, hypergeometric functions, are a fascinating mixture of algebra, analysis and combinatorics, and among the most ubiquitous mathematical objects. They seem to occur naturally almost everywhere — following are just a few examples to illustrate this. If you try to solve the Laplace partial differential equation by separation of variables, the Bessel equation appears naturally: its solutions are hypergeometric [SD64]. When parameterizing elliptic curves, one encounters theta functions, which are hypergeometric [Yos97]. Perhaps one is trying to solve a polynomial equation of degree n in terms of the coefficients: radicals will not be enough to do this if $n > 4$, but hypergeometric functions will [Stu00]. Or maybe you want to do least squares approximations on sets of data, and the polynomial basis you need to use involves orthogonal polynomials; all interesting such bases consist of hypergeometric elements [KS]. In mirror symmetry, the periods of certain natural differentials in families of Calabi–Yau toric hypersurfaces satisfy hypergeometric equations [CK99]. If you want to count combinatorial objects and your quantities satisfy recursions, then this often forces their generating function to be hypergeometric. In a recent instance of this phenomenon involving algebraic geometry, the generating functions for intersection numbers on moduli spaces of curves turn out to be A-hypergeometric in the sense of Gelfand, Graev, Kapranov and Zelevinsky [Oko02]. It is this A-hypergeometric approach that we shall follow in this article.

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Gelfand, Graev and Zelevinsky defined A-hypergeometric systems in the mid-eighties, and they were further developed by Gelfand, Kapranov and Zelevinsky (see [GGZ87, GZK89, GZK93]). Before we give the general definition of A-hypergeometric systems, let us consider one example.

Example 1.1. Let A be the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$. We consider the integral kernel $\ker_{\mathbb{Z}}(A)$ of A consisting of all $u \in \mathbb{Z}^3$ with $A \cdot u = 0$. For our A we have that $\ker_{\mathbb{Z}}(A)$ is generated by $u = (1, -2, 1)$. We use this vector to form the operator $\Delta(u) = \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_3} - \frac{\partial^2}{\partial {x_2}^2}$ by separating the positive part $u_+ = (1, 0, 1)$ from the negative part $u_- = (0, 2, 0)$ of u and then using the entries as exponents over the corresponding derivations.

From the two rows of the matrix we create the operators

$$\begin{aligned} E_1 &= 1 \cdot x_1 \frac{\partial}{\partial x_1} + 1 \cdot x_2 \frac{\partial}{\partial x_2} + 1 \cdot x_3 \frac{\partial}{\partial x_3}, \\ E_2 &= 2 \cdot x_1 \frac{\partial}{\partial x_1} + 1 \cdot x_2 \frac{\partial}{\partial x_2} + 0 \cdot x_3 \frac{\partial}{\partial x_3}. \end{aligned}$$

For any pair $\beta = (\beta_1, \beta_2)$ of complex numbers, the A-hypergeometric system is the system of linear partial differential equations

$$\begin{aligned} E_1 \bullet (\varphi) &= \beta_1 \cdot \varphi, \\ E_2 \bullet (\varphi) &= \beta_2 \cdot \varphi, \\ \left(\frac{\partial^2}{\partial x_1 \partial x_3} - \frac{\partial^2}{\partial {x_2}^2} \right) \bullet (\varphi) &= 0 \end{aligned} \tag{1}$$

where φ is a function in the three variables x_1, x_2, x_3. One may interpret (β_1, β_2) as a multi-degree of the solution φ as we explain now. First notice that:

$$\left(x_i \frac{\partial}{\partial x_i} \right) \bullet ({x_1}^{\alpha_1} {x_2}^{\alpha_2} {x_3}^{\alpha_3}) = \alpha_i {x_1}^{\alpha_1} {x_2}^{\alpha_2} {x_3}^{\alpha_3} \;, \quad i = 1, 2, 3.$$

This means, using linearity, that for a power series $\varphi(x_1, x_2, x_3) = \sum_\alpha c_\alpha {x_1}^{\alpha_1} {x_2}^{\alpha_2} {x_3}^{\alpha_3}$ we have:

$$\begin{aligned} (E_1 - \beta_1) \bullet \varphi &= \sum_\alpha c_\alpha (E_1 - \beta_1) \bullet \left({x_1}^{\alpha_1} {x_2}^{\alpha_2} {x_3}^{\alpha_3} \right) \\ &= \sum_\alpha c_\alpha \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - \beta_1 \right) \bullet \left({x_1}^{\alpha_1} {x_2}^{\alpha_2} {x_3}^{\alpha_3} \right) \\ &= \sum_\alpha c_\alpha \left(\alpha_1 + \alpha_2 + \alpha_3 - \beta_1 \right) {x_1}^{\alpha_1} {x_2}^{\alpha_2} {x_3}^{\alpha_3}. \end{aligned}$$

Thus, if $(E_1 - \beta_1) \bullet \varphi = 0$, then the exponents α appearing in $\varphi = \sum_\alpha c_\alpha {x_1}^{\alpha_1} {x_2}^{\alpha_2} {x_3}^{\alpha_3}$ must satisfy:

$$[c_\alpha \neq 0] \Longrightarrow [\alpha_1 + \alpha_2 + \alpha_3 = \beta_1] \,.$$

A similar computation using E_2 instead of E_1 yields:

$$[c_\alpha \neq 0] \Longrightarrow [2\alpha_1 + \alpha_2 = \beta_2]$$

and the two implications combine to

$$[c_\alpha \neq 0] \Longrightarrow [A \cdot \alpha = \beta] \,. \tag{2}$$

Let us define the *multi-degree* of x_i to be the ith column of A:

$$\deg(x_1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix};\ \deg(x_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix};\ \deg(x_3) = \begin{pmatrix} 1 \\ 0 \end{pmatrix};$$

hence the multi-degree of a monomial is given by:

$$\deg({x_1}^{\alpha_1}{x_2}^{\alpha_2}{x_3}^{\alpha_3}) = A\cdot\alpha.$$

Now equation (2) translates into:

$$\text{If } \varphi = \textstyle\sum_\alpha c_\alpha {x_1}^{\alpha_1}{x_2}^{\alpha_2}{x_3}^{\alpha_3} \text{ is killed by } E_1-\beta_1 \text{ and } E_2-\beta_2, \text{ then}$$
$$[c_\alpha \neq 0] \Longrightarrow [\deg({x_1}^{\alpha_1}{x_2}^{\alpha_2}{x_3}^{\alpha_3}) = \beta].$$

To illustrate one point made in the introduction above, let $\beta = (0,-1)$. It is well-known and easy to verify that then the two roots $z_{1,2} = \frac{-x_2\pm\sqrt{{x_2}^2-4x_1x_3}}{2x_1}$ of the polynomial $x_1z^2+x_2z+x_3$ in the variable z with indeterminate coefficients x_1, x_2, x_3 are solutions of the system (1). In turn, one can use the system of partial differential equations to obtain a formula of the roots as a hypergeometric series:

$$z_{1,2} = \frac{-x_2}{2x_1} \pm \left(\frac{x_2}{2x_1} - \frac{x_3}{x_2}\sum_{t=0}^{\infty}\frac{1}{t+1}\binom{2t}{t}\left(\frac{x_1x_3}{{x_2}^2}\right)^t\right).$$

We now come to the definition of a general A-hypergeometric system. We begin with taking an integer $d\times n$ matrix $A = (a_{i,j})$ of full rank d and a complex parameter vector β. As in the example we form for $1\leq i\leq d$ the operators

$$E_i = \sum_{j=1}^{n} a_{i,j}\, x_j \frac{\partial}{\partial x_j}$$

from the rows of A.

Definition 1.2. The *A-hypergeometric system with parameter β*, denoted $H_A(\beta)$, is the following system of linear partial differential equations with polynomial coefficients for the function $\varphi = \varphi(x_1,\ldots,x_n)$:

$$E_i \bullet (\varphi) = \beta_i\cdot\varphi \qquad\qquad i = 1,\ldots d;$$

$$\left(\prod_{u_i>0}\frac{\partial^{u_i}}{\partial {x_i}^{u_i}}\right)\bullet(\varphi) = \left(\prod_{u_i<0}\frac{\partial^{-u_i}}{\partial {x_i}^{-u_i}}\right)\bullet(\varphi) \qquad \text{for all } u\in\ker_{\mathbb{Z}}(A).$$

The first d equations above are called *homogeneity conditions*, the remaining equations are called *toric equations*.

For notational convenience we shall from now on abbreviate the derivation $\frac{\partial}{\partial x_i}$ by simply ∂_i. Then $R_A = \mathbb{C}[\partial_1,\ldots,\partial_n]$ is the ring of $\mathbb{C}$-linear differential operators with constant coefficients. Let us view Example 1.1 in the light of our definition of general hypergeometric systems. In Definition 1.2 there are infinitely many toric equations, one for each element u of $\ker_{\mathbb{Z}}(A)$. On the other hand, in (1) we listed only one such, $\Delta(u)\bullet\varphi = 0$ with $u = (1,-2,1)$. Yet it turns out that no information is lost. Namely, if A is the matrix of

Example 1.1 and $v \in \ker_{\mathbb{Z}}(A)$ then up to sign $v = (k, -2k, k)$ for some natural number k. It follows that, again up to sign,

$$\begin{aligned}\Delta(v) &= (\partial_1\partial_3)^k - {\partial_2}^{2k} \\ &= \left((\partial_1\partial_3)^{k-1} + (\partial_1\partial_3)^{k-2}{\partial_2}^2 + (\partial_1\partial_3)^{k-3}{\partial_2}^4 + \cdots + {\partial_2}^{2k-2}\right) \cdot \left(\partial_1\partial_3 - {\partial_2}^2\right).\end{aligned}$$

So if φ is annihilated by $\Delta(u)$ then it is also annihilated by $\Delta(v)$ for all other $v \in \ker_{\mathbb{Z}}(A)$.

More generally, it turns out that for any matrix A one always only needs to look at a finite number of toric equations; in order to explain the reasons for this we simplify our notation a bit as follows. In the remainder of the paper we would like to use multi-index notation: if $u \in \mathbb{Z}^n$ we mean by x^u the (Laurent) monomial ${x_1}^{u_1}{x_2}^{u_2}\cdots{x_n}^{u_n}$; a similar convention shall be used for ∂^u. Also, if $u \in \mathbb{Z}^n$, we write $u = u_+ - u_-$, where:

$$(u_+)_i = \max\{u_i, 0\}, \quad (u_-)_i = \max\{-u_i, 0\}.$$

With this notation, the toric operator $\Delta(u) = \prod_{u_i>0} \frac{\partial^{u_i}}{\partial {x_i}^{u_i}} - \prod_{u_i<0} \frac{\partial^{-u_i}}{\partial {x_i}^{-u_i}}$ in $H_A(\beta)$ corresponding to $u \in \ker_{\mathbb{Z}}(A)$ becomes $\partial^{u_+} - \partial^{u_-}$. Let I_A be the *toric ideal* in R_A generated by all $\Delta(u) = \partial^{u_+} - \partial^{u_-}$ with $u \in \ker_{\mathbb{Z}}(A)$. Since R_A is Noetherian, there is a finite set of generators for this ideal. In fact, since I_A is generated by *binomials*, this finite generating set will consist of binomials and hence be of the form $\{\Delta(v_1), \ldots, \Delta(v_k)\}$ for some elements $v_1, \ldots, v_k$ in $\ker_{\mathbb{Z}}(A)$. Indeed, there are simple algorithms to find such a collection $\{v_i\}_{i=1}^k$, see [Stu96].

Although we will not use this, we would like to mention that by a theorem of Stafford [Sta78] the entire A-hypergeometric system is equivalent to a linear system of just *two* differential equations. However, these two equations are very complicated since they have to carry a lot of information.

Since $H_A(\beta)$ is a *linear* system of equations, the set of its holomorphic solutions on a simply connected open set in $\mathbb{C}^n$ forms a vector space over the complex numbers. The dimension of this vector space we shall call the *rank* of $H_A(\beta)$ and denote it by $\operatorname{rank}(H_A(\beta))$. Somewhat surprisingly, the rank turns out to be finite for any choice of A and β — this is a highly unusual event for systems partial differential equations.

So one of the most basic questions one might ask about the A-hypergeometric system $H_A(\beta)$ is:

Question A: What is the rank of $H_A(\beta)$?

A first answer to this question was given by Gelfand, Kapranov and Zelevinsky [GZK89, GZK93] who found that under a certain condition on the ideal I_A called *Cohen–Macaulayness*, $\operatorname{rank}(H_A(\beta))$ is actually *independent* of β. To describe this condition, consider the polynomial ring $R_A = \mathbb{C}[\partial_1, \ldots, \partial_n]$ from above and its quotient $S_A = R_A/I_A$. Then one calls I_A *Cohen–Macaulay* if and only if there are $d = \operatorname{rank}(A)$ linear forms $L_1, \ldots, L_d$ in R_A such that for all $1 \leq i \leq d$ the form L_i is a non-zerodivisor on $S_A/\langle L_1, \ldots, L_{i-1}\rangle$. This property is a way of allowing singularities to occur in S_A while preserving many good algebraic properties. By a theorem of Hochster [Hoc72], one particular class of Cohen–Macaulay examples is provided by those matrices A for which the collection $\mathbb{N}A$ of all $\mathbb{N}$-linear combinations of the columns of A is *saturated*. This condition means that if a lattice point $p \in \mathbb{Z}^d$ has some multiple $p + \cdots + p$ in $\mathbb{N}A$, then p itself is already in $\mathbb{N}A$. Such saturated semigroups arise naturally as the collection of all lattice points inside the *positive cone* $\mathbb{R}_+v_1 + \cdots + \mathbb{R}_+v_k$ of k lattice points $v_1, \ldots, v_k \in \mathbb{Z}^d$. Our Example 1.1 is of this type with $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Under the assumption of Cohen–Macaulayness, a completely explicit combinatorial formula for the rank was provided in [GZK89, Ado94]. Let us describe this formula. Form a polytope Q_0 by taking the convex hull of the columns of A and the origin, pictured as points in $\mathbb{R}^d$. Since A has full rank this polytope has dimension d. Then the *simplicial* or *normalized volume* of A, denoted by $\mathrm{vol}(A)$ equals the product of $d!$ and the usual Euclidean volume of Q_0 (so that, for example, a standard d-simplex has simplicial volume equal to 1). With this notation, if I_A is Cohen–Macaulay, then the rank $\mathrm{rank}(H_A(\beta))$ of the hypergeometric system to A and β agrees with the simplicial volume $\mathrm{vol}(A)$ no matter what the parameter $\beta \in \mathbb{C}^d$ is.

Several authors have expanded on these results, usually in the *homogeneous* case where all the columns of A, considered as points in $\mathbb{R}^d$, lie in a hyperplane not containing the origin. For example, Adolphson [Ado94] showed that even if A fails to be Cohen–Macaulay then the formula $\mathrm{rank}(H_A(\beta)) = \mathrm{vol}(A)$ is valid for *almost every* β. If A is homogeneous, but under no other conditions on either A or β, we always have $\mathrm{rank}(H_A(\beta)) \geq \mathrm{vol}(A)$ as was shown by Saito, Sturmfels and Takayama [SST00]. Considering these results, the natural question is:

Question B: Are there actually any examples where $\mathrm{rank}(H_A(\beta)) > \mathrm{vol}(A)$?

The answer is "yes", and the first and smallest example of this type was given in [ST98]; we will revisit it in Example 2.1. Experimental studies showed that constructing rank-jumping examples (A, β) is very hard since they are quite rare; this accounts for the 10-year delay between the first results on A-hypergeometric functions and the discovery of the first rank-jump.

One reason that makes rank-jumps very interesting is that they seem to coincide with the existence of very nice solutions: contrary to typical solutions which are proper power series, in all cases that are known to the authors the "extra" solutions at a rank-jump are *Laurent polynomials* (or Puiseux polynomials, if the exponents are non-integral); this fact is not well understood yet. Viewing the results of [Ado94, GZK89, SST00] in the light of Example 2.1, one is then lead to three more precise questions:

Questions C:

(1) Which matrices A allow for rank jumps?
(2) If A has a rank jump at all, which parameters are rank-jumping?
(3) If β is a rank-jumping parameter for A, by how much does the rank exceed the volume?

The first two questions have been recently answered in full [MMW04]. In the present article we are interested in the third question and investigate the possible magnitude of the gap between rank and volume. There is a known upper bound for the rank in terms of the volume given by $\mathrm{rank}(H_A(\beta)) \leq 2^{2d} \cdot \mathrm{vol}(A)$, see [SST00, Corollary 4.1.2]. It is believed that this exponential upper bound is not optimal. In fact, until now no example had been known in which the rank exceeds the volume by three or more.

The goal of this article to describe a family of examples that exhibit arbitrarily large rank jumps, we shall prove:

Theorem 1.3. *For any $d \in \mathbb{Z}_{>1}$ there exists a $d \times (2d)$-matrix A_d and a parameter $\beta_d \in \mathbb{C}^d$ such that*

$$\mathrm{rank}(H_{A_d}(\beta_d)) - \mathrm{vol}(A_d) \geq d - 1.$$

In contrast to the substantial amount of algebra and analysis that is needed to prove most of the results quoted above, the proof of our result is completely elementary, requires only a knowledge of linear algebra and is based on constructing Laurent polynomial solutions.

2. THE FIRST RANK-JUMP EXAMPLE

We now present a major player in our later constructions: the first ever rank-jumping example.

Example 2.1. Let $\beta = (\beta_1, \beta_2)$ and

$$A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}.$$

Then I_{A_2} is generated by

$$\begin{array}{rcl} \partial_2\partial_3 & - & \partial_1\partial_4, \\ \partial_1^2\partial_3 & - & \partial_2^3, \\ \partial_2\partial_4^2 & - & \partial_3^3, \\ \partial_1\partial_3^2 & - & \partial_2^2\partial_4 \end{array}$$

and there are two homogeneity conditions:

$$\begin{array}{rcl} (x_1\partial_1 + x_2\partial_2 + \;\; x_3\partial_3 + \;\; x_4\partial_4 - \beta_1) \bullet (\varphi) & = & 0, \\ (x_2\partial_2 + 3x_3\partial_3 + 4x_4\partial_4 - \beta_2) \bullet (\varphi) & = & 0. \end{array}$$

In this case,

$$\operatorname{rank}(H_{A_2}(\beta_1, \beta_2)) = \begin{cases} 4 = \operatorname{vol}(A_2) & \text{if } (\beta_1, \beta_2) \neq (1, 2), \\ 5 & \text{if } (\beta_1, \beta_2) = (1, 2). \end{cases}$$

Example 2.1 was completely analyzed in [ST98]. We refer to that article for a proof that $(1, 2)$ is indeed the unique parameter for which rank exceeds volume. We now present an explicit basis for the solution space of $H_{A_2}(1, 2)$.

Theorem 2.2 (Proposition 4.1 [ST98])**.** *Let*

$$u^{(1)} = (1/2, 0, 0, 1/2), \quad u^{(2)} = (1/4, 1, 0, 1/4), \quad u^{(3)} = (1/4, 0, 1, -1/4)$$

and put for $i = 1, 2, 3$

$$\Omega_i = \left\{ (a, b) \in \mathbb{Z}^2 : u_2^{(i)} + 4a \geq 3b,\ u_3^{(i)} + b \geq 0 \right\}.$$

Consider for $i = 1, 2, 3$ *the functions*

$$f_i = \sum_{(a,b)\in\Omega_i} c_{a,b}\, x^{u^{(i)}+a(-3,4,0,-1)+b(2,-3,1,0)}$$

where

$$c_{a,b} = \frac{1}{\Gamma(u_1^{(i)} - 3a + 2b + 1)\Gamma(u_2^{(i)} + 4a - 3b + 1)\Gamma(u_3^{(i)} + b + 1)\Gamma(u_4^{(i)} - a + 1)}$$

and Γ *denotes the usual gamma function. If one sets*

$$p_1 = \frac{{x_2}^2}{x_1}, \qquad p_4 = \frac{{x_3}^2}{x_4}$$

then the five functions p_1, p_4, f_1, f_2, f_3 *are a basis for the solution space of* $H_{A_2}(1, 2)$. $\square$

3. CONSTRUCTING ARBITRARY JUMPS

We are now ready to provide, for given $d \geq 2$, a $d \times 2d$ matrix A_d and a parameter $\beta_d \in \mathbb{N}^d$ such that

$$\operatorname{rank}(H_{A_d}(\beta_d)) \geq \operatorname{vol}(A_d) + d - 1.$$

As we mentioned before, previously no example existed where the gap between rank and volume exceeds two.

If $d = 2$, Example 2.1 will do. So for the remainder of this article we fix an integer $d \geq 3$, and we write A and β instead of A_d and β_d in order to simplify notation.

Let $e_1, \dots, e_d$ be the standard basis vectors in $\mathbb{C}^d$. Define $a_1, \dots, a_{2d} \in \mathbb{N}^d$ as follows:

$$\begin{aligned} a_1 &= (1, 0, \dots, 0, 0), \\ a_2 &= (1, 0, \dots, 0, 1), \\ a_3 &= (1, 0, \dots, 0, 3), \\ a_4 &= (1, 0, \dots, 0, 4), \end{aligned}$$

while if $3 \leq k \leq d - 1$, set

$$a_{2k-1} = e_1 + e_{k-1}, \qquad a_{2k} = e_1 + e_{k-1} + e_d.$$

Thus

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & & 0 & 0 \\ \vdots & \vdots & & & & & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 1 & 1 \\ 0 & 1 & 3 & 4 & 0 & 1 & 0 & 1 & \cdots & 0 & 1 \end{pmatrix}$$

Now let

$$\beta = (1, 0, \dots, 0, 2).$$

We shall prove

Theorem 3.1. *For the matrix A and parameter β introduced above, we have:*

$$\operatorname{rank}(H_A(\beta)) - \operatorname{vol}(A) \geq d - 1.$$

We will prove this theorem in a series of lemmas. First we will compute the simplicial volume $\operatorname{vol}(A)$; after this is done, we will exhibit the required number of linearly independent solutions of $H_A(\beta)$.

Lemma 3.2. *The simplicial volume of A is $d + 2$.*

Proof. Let $Q = \operatorname{conv}(A)$, the convex hull of the columns of A. Since the columns of A all lie in the hyperplane $t_1 = 1$ of $\mathbb{R}^d$, the convex hull Q_0 of the origin and the columns of A form a pyramid of height one over Q. Hence the simplicial volume of Q_0 is equal to the simplicial volume of Q; we compute the latter.

The polytope Q is the union of two others: the prism P (over the standard $(d-2)$-simplex with vertices $p_1, p_5, p_7, \dots, p_{2d-1}$) whose vertices are the columns of:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & & 0 & 0 \\ \vdots & & & & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & \cdots & 0 & 1 \end{pmatrix},$$

and the $(d-1)$-simplex S whose vertices $p_2, p_4, p_6, \dots, p_{2d}$ are the columns of:

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ 1 & 4 & 1 & \cdots & 1 \end{pmatrix}.$$

In Figure 1 we see the decomposition of Q into the prism P and the simplex S for $d = 4$. Since the prism

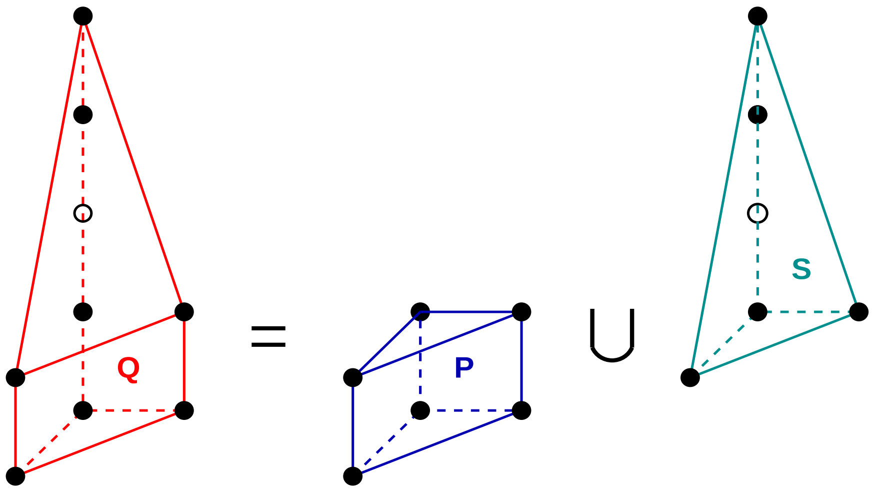

FIGURE 1. Decomposing $Q = \mathrm{conv}(A)$ for $d = 4$

has height one, its Euclidean volume equals the Euclidean volume of its base, the standard $(d-2)$-simplex with Euclidean volume $\frac{1}{(d-2)!}$. Thus, P has simplicial volume $\frac{(\dim(P))!}{(d-2)!} = d - 1$.

On the other hand, S is a pyramid of height three over a standard simplex, and so its simplicial volume is 3. This implies that $\mathrm{vol}(Q) = (d-1) + 3 = d + 2$. □

The next step in our proof is to construct $2d+1$ solutions of $H_A(\beta)$. In order to do this we need to understand the integer kernel of A, because the toric equations are constructed directly from these elements. In particular, we will identify positive and negative coordinates of certain elements in $\ker_{\mathbb{Z}}(A)$. The other important ingredient is finding integer solutions of $A \cdot u = \beta$. The fact that the coordinates of β are small

positive integers will facilitate this search. However, we start with showing that any solution of $H_{A_2}(1,2)$ is a solution of our system.

Lemma 3.3. *Let ψ be a solution of $H_{A_2}(1,2)$. Then ψ is a solution of $H_A(\beta)$. In particular, the functions p_1, p_4, and f_1, f_2, f_3 from Theorem 2.2 are linearly independent solutions of $H_A(\beta)$.*

Proof. It is easy to see that ψ is a solution of the homogeneity equations

$$\sum_{j=1}^{2d} a_{i,j}\, x_j \partial_j \bullet (\psi) = \beta_i \cdot \psi, \qquad\qquad i = 1, \dots, d.$$

Hence we only need to verify that ψ is annihilated by the toric operators $\Delta(u) = \partial^{u_+} - \partial^{u_-}$ for all $u_+ - u_- = u \in \ker_{\mathbb{Z}}(A)$. We now study the integer kernel A. Since A is of full rank d and the columns of the following $(2d \times d)$-matrix B are linearly independent, the columns of B form a basis for the kernel of B over the rational numbers:

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ -2 & -1 & -1 & -1 & -1 & \cdots & -1 \\ 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & -1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Using rows $1, 2, 5, 7, 9, \dots, 2d-1$ we see that the greatest common divisor of the maximal minors of B is 1. This implies that the columns of B are actually a basis for the integer kernel $\ker_{\mathbb{Z}}(A)$: any element of $\ker_{\mathbb{Z}}(A)$ is an *integer* linear combination of the columns of B.

Choose a toric operator $\partial^{u_+} - \partial^{u_-}$ where $u = B \cdot z$ for some $z \in \mathbb{Z}^d$. If $z_i \neq 0$ for some $i \geq 3$, then u_{2i-1} and u_{2i} will be nonzero, with opposite signs. This means that one of the monomials in $\partial^{u_+} - \partial^{u_-}$ will contain ∂_{2i-1}, and the other will contain ∂_{2i}. Since ψ does not contain the variables x_{2i-1} nor x_{2i}, it follows that both monomials annihilate ψ and therefore $(\partial^{u_+} - \partial^{u_-}) \bullet (\psi) = 0$.

It remains to consider the case when only z_1 and z_2 are allowed to be nonzero. But in that case $u = B \cdot z$ gives a toric operator inside $H_{A_2}(1,2)$, and ψ was assumed to be a solution of that system. □

We note that there are no polynomial solutions for $H_A(\beta)$ since any such solution would have to have multi-degree β and β is not an $\mathbb{N}$-linear combination of the multi-degrees of the x_i, which are the columns of A. We will now construct Laurent polynomial solutions for $H_A(\beta)$, one for each vertex of the polyhedron $Q = \operatorname{conv}(A)$. These vertices are the columns a_1, a_4, $a_5, \dots, a_{2d}$ of the matrix A. The correspondence between the Laurent polynomials p_i and the vertices a_i will be given by

p is associated to a_i if no variable but x_i occurs in any denominator of p.

For the vertices a_1 and a_4 we already have such solutions, namely the Laurent monomials $p_1 = {x_2}^2/x_1$ and $p_4 = {x_3}^2/x_4$. So we need to construct Laurent polynomial solutions p_i of $H_A(\beta)$ associated to $a_5, \ldots, a_{2d}$, and since $H_A(\beta)$ does not have polynomial solutions, these are proper fractions.

If $p = \sum c_\alpha x^\alpha$ is a Laurent polynomial solution of $H_A(\beta)$ then the homogeneity equations imply that $A \cdot \alpha = \beta$ for any α such that $c_\alpha \neq 0$. Hence the possible Laurent monomials appearing in a Laurent solution p_i of $H_A(\beta)$ associated to a_i are of the form x^α where $A \cdot \alpha = \beta$, $\alpha \in \mathbb{Z}^n$ and only α_i is a negative integer.

Let us search for all such vectors α when $i = 5$. Since the second coordinate of β is zero and only the columns a_5 and a_6 of A have nonzero second coordinates, we must have $\alpha_6 = -\alpha_5 > 0$.

Then

$$\alpha_5 a_5 + \alpha_6 a_6 = \alpha_6 e_d.$$

Note that A has no negative entries. As $\alpha_i \geq 0$ for $i \neq 5$, $A \cdot \alpha = (1, 0, \ldots, 0, 2)$ is in each component bounded from below by $\alpha_5 a_5 + \alpha_6 a_6$, so α_6 equals 1 or 2. Moreover, every a_i has a 1 in the first coordinate and so α has precisely one more nonzero entry besides α_5 and α_6; this entry will be a 1. Now if $\alpha_j = 1$ for any $j > 6$ then $A \cdot \alpha$ will have a 1 in a place where β has a zero. Therefore, the third nonzero coordinate of α must be one of α_1, α_2, α_3 or α_4. If $\alpha_6 = 1$, we get $\alpha = (0, 1, 0, 0, -1, 1, 0, 0, \ldots, 0)$ while for $\alpha_6 = 2$ we get $\alpha = (1, 0, 0, 0, -2, 2, 0, 0, \ldots, 0)$; there is no other choice.

This gives us two possible monomials to make a Laurent polynomial solution of $H_A(\beta)$ where only x_5 is in the denominator, namely the monomials $\frac{x_2 x_6}{x_5}$ and $\frac{x_1 {x_6}^2}{{x_5}^2}$. Neither of these Laurent monomials is a solution for $H_A(\beta)$, but a suitable linear combination is:

Lemma 3.4. *The function*

$$p_5 = \frac{x_2 x_6}{x_5} - \frac{1}{2} \frac{x_1 {x_6}^2}{{x_5}^2}$$

is a solution of $H_A(\beta)$.

Proof. By our construction, p_5 is a solution of the homogeneity equations,

$$\sum_{j=1}^{2d} a_{i,j}\, x_j \partial_j \bullet (\psi) = \beta_i \cdot \psi, \qquad i = 1, \ldots, d,$$

because the exponents appearing in it satisfy $A \cdot \alpha = \beta$. Now we need to see that p_5 is a solution to

$$(\partial^{u_+} - \partial^{u_-}) \bullet (p_5) = 0$$

whenever $u_+ - u_- = u \in \ker_{\mathbb{Z}}(A)$.

Recall that $\ker_{\mathbb{Z}}(A)$ has a $\mathbb{Z}$-basis consisting of the columns of the matrix B. Let us look at a toric equation $\partial^{u_+} - \partial^{u_-}$, where $u = B \cdot z$ for some integer vector $z \in \mathbb{Z}^d$. If $z_i \neq 0$ for some $i > 3$, then u_{2i-1} and u_{2i} are nonzero with opposite signs. Then ∂_{2i-1} and ∂_{2i} appear in different monomials in $\partial^{u_+} - \partial^{u_-}$ while p_5 does not contain either of the variables x_{2i-1} or x_{2i}. This means that

$$(\partial^{u_+} - \partial^{u_-}) \bullet (p_5) = 0 \quad \text{for } u_+ - u_- = B \cdot z \text{ with } z_i \neq 0 \text{ for some } i > 3.$$

So let us now look at $u = B \cdot z$ for z such that $z_i = 0$, $i = 4, 5, \ldots, d$. Then the only (possibly) nonzero coordinates of u are the following:

$$\begin{aligned}
u_1 &= z_1 + z_2 + z_3, \\
u_2 &= -2z_1 - z_2 - z_3, \\
u_3 &= 2z_1 - z_2, \\
u_4 &= -z_1 + z_2, \\
u_5 &= -z_3, \\
u_6 &= z_3,
\end{aligned}$$

with all $z_i \in \mathbb{Z}$. If u_3 and u_4 are both nonzero and have different signs, then the fact that p_5 contains neither x_3 nor x_4 implies that $(\partial^{u_+} - \partial^{u_-}) \bullet (p_5) = 0$. This means that we need to study three cases:

(1) $u_3 = u_4 = 0$,
(2) $0 \leq u_3, u_4$ and not both u_3 and u_4 vanish,
(3) $0 \geq u_3, u_4$ and not both u_3 and u_4 vanish.

In Case (1), we have $z_1 = z_2 = 0$. If $|z_3| \geq 2$, then we have ∂_1^2 and ∂_2^2 in different monomials of $\partial^{u_+} - \partial^{u_-}$, which implies that $(\partial^{u_+} - \partial^{u_-}) \bullet (p_5) = 0$. In the remaining case $|z_3| = 1$ one finds

$$(\partial_1\partial_6 - \partial_2\partial_5) \bullet (p_5) = 0 - \frac{1}{2}\frac{2x_6}{{x_5}^2} - \frac{-x_6}{{x_5}^2} - 0 = 0.$$

In Case (2) one sees immediately that ∂^{u_+} kills p_5 since u_3 or u_4 will be positive and p_5 does not involve either variable. So we need to show that ∂^{u_-} also kills p_5. From the given inequalities one deduces that either $z_1 = z_2 = 1$ or that $z_1 \geq 1$ and $z_2 \geq 2$. In the latter situation $u_2 \leq -4 - z_3 \leq -2$, so ∂^{u_-} contains ${\partial_2}^2$ and hence kills p_5. We now consider the case $z_1 = z_2 = 1$. Clearly if $z_3 < -2$ then ∂^{u_-} contains ${\partial_6}^3$ and hence kills p_5. If $z_3 = -2$ then $\partial^{u_-} = \partial_2{\partial_6}^2$ kills p_5. Finally, if $z_3 \geq -1$ then ∂^{u_-} contains ${\partial_2}^2$ and kills p_5.

Case (3) is entirely parallel to Case (2), with signs reversed. □

The construction of p_6 goes along the same lines as the construction of p_5. First we find that the only solutions of $A \cdot \alpha = \beta$ with $\alpha_i \in \mathbb{Z}_{\geq 0}$ for $i \neq 6$ and $\alpha_6 \in \mathbb{Z}_{<0}$ are the vectors

$$(0, 0, 1, 0, 1, -1, 0, \ldots, 0) \quad \text{and} \quad (0, 0, 0, 1, 2, -2, 0, \ldots, 0),$$

Then we propose

$$p_6 = \frac{x_3x_5}{x_6} - \frac{1}{2}\frac{x_4{x_5}^2}{{x_6}^2}.$$

A similar analysis as in Lemma 3.4 shows that, except for $\partial_3\partial_6 - \partial_4\partial_5$, every generator $\partial^{u_+} - \partial^{u_-}$ of I_A has the property that both ∂^{u_+} and ∂^{u_-} annihilate p_6. Now to establish p_6 as solution of $H_A(\beta)$ reduces to checking that

$$(\partial_3\partial_6 - \partial_4\partial_5) \bullet (p_6) = \frac{-x_5}{{x_6}^2} - \frac{-1}{2}\frac{2x_5}{{x_6}^2} = 0.$$

More generally, adapting the notation, we obtain:

Proposition 3.5. *The two functions*

$$p_{2i-1} = \frac{x_2 x_{2i}}{x_{2i-1}} - \frac{1}{2}\frac{x_1 {x_{2i}}^2}{{x_{2i-1}}^2}, \qquad p_{2i} = \frac{x_3 x_{2i-1}}{x_{2i}} - \frac{1}{2}\frac{x_4 {x_{2i-1}}^2}{{x_{2i}}^2}$$

are solutions of $H_A(\beta)$ for every integer i with $3 \leq i \leq d$.

We can now complete the proof of our main result.

Proof of Theorem 3.1. The functions f_1, f_2, f_3 and $p_1, p_4, p_5, \ldots, p_{2d}$ are $2d + 1$ solutions of $H_A(\beta)$. By Theorem 2.2, the first five are linearly independent. Since for $i > 4$ the Laurent solution p_i has a pole in x_i and since x_i does not occur in the solutions $f_1, f_1, f_3, p_1, p_4, \ldots, p_{i-1}$ we conclude that all these solutions are linearly independent. It follows that $\operatorname{rank}(H_A(\beta)) \geq 2d + 1$. Using $\operatorname{vol}(A) = d + 2$, we conclude that

$$\operatorname{rank}(H_A(\beta)) - \operatorname{vol}(A) \geq d - 1,$$

which is what we wanted to prove. □

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(Laura Felicia Matusevich) DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA
E-mail address: `laura@math.harvard.edu`

(Uli Walther) DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, IN
E-mail address: `walther@math.purdue.edu`